### Lecture 4

Logical Equivalence, Predicates and Quantifiers

# Logical Equivalences with $\rightarrow$ and $\leftrightarrow$

$$p \rightarrow q \equiv \neg p \lor q$$

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

• 
$$(p \to q) \land (p \to r) \equiv p \to (q \land r)$$

$$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$$

$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

$$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$$

# Logical Equivalences

Why logical equivalence laws hold for general compound propositions as well?

### **Example:**

$$\neg (p \lor q) \equiv \neg p \land \neg q$$

$$Y$$

$$\overline{\neg ((r \lor \neg (s \land t)) \lor (t \to (u \leftrightarrow r)))} \equiv \overline{\neg (r \lor \neg (s \land t)) \land \neg (t \to (u \leftrightarrow r))}$$

Suppose for r = T, s = F, t = T, and u = T:

- Truth value of Y and Z differs.
- $rac{1}{2}$   $rac{1}$   $rac{1}{2}$   $rac{1}$   $rac{1}{2}$   $rac{1}$   $rac{$

implies 
$$\neg (W \lor X) \neq \neg W \land \neg X$$
 (not possible due to De Morgan's law)

## Proving Logical Equivalences

Proving logical equivalence of two propositions using truth table is time-consuming.

Logical equivalence can also be proven using existing laws of logical equivalences.

Example: Prove 
$$\neg (p \lor (\neg p \land q)) \equiv \neg p \land \neg q$$

$$\neg (p \lor (\neg p \land q)) \equiv \neg p \land \neg (\neg p \land q) \qquad (by \ De \ Morgan's \ law)$$

$$\equiv \neg p \land (\neg (\neg p) \lor \neg q) \qquad (by \ De \ Morgan's \ law)$$

$$\equiv \neg p \land (p \lor \neg q) \qquad (by \ double \ negation \ law)$$

$$\equiv (\neg p \land p) \lor (\neg p \land \neg q) \qquad (by \ distributive \ law)$$

$$\equiv F \lor (\neg p \land \neg q) \qquad (by \ negation \ law)$$

$$\equiv (\neg p \land \neg q) \lor F \qquad (by \ commutative \ law)$$

$$\equiv \neg p \land \neg q \qquad (by \ identity \ law)$$

# Predicate Logic

Let's revisit an old example.

```
Assumptions: \( \) 1. All men are mortal. \( 2. \) Socrates is a man.
```

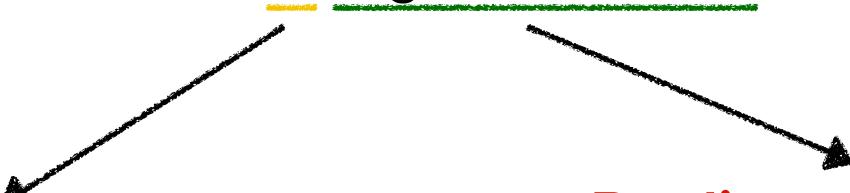
Conclusion: Socrates is mortal.

Does the conclusion follow from assumptions using rules of propositional logic? No, it seems.

We need more powerful form of logic called predicate logic to express these reasonings.

## Predicates

Statement "x is greater than 3" has two parts.



**Subject** of the statement: "x"

Predicate of the statement: "is greater than 3"

(Refers to a property that subject can have.)

#### **Convention:**

P(x) = "x is greater than 3", where P denotes the predicate and x denotes the subject.

P(x) is called propositional function and becomes a proposition when x is assigned a value.

### **Example:**

Q(x, y) = x is a factor of y

Q(4,100) = has truth value 'true', while Q(3,25) has truth value 'false'.

## Quantifiers

Quantification expresses the extent to which a predicate is true over a range of elements.

#### **Universal Quantification:**

For P(x), it conveys that P(x) is true for all values of x from a certain domain.

#### **Existential Quantification:**

For P(x), it conveys that P(x) is true for some value of x from a certain domain.

Domain is important.

## Universal Quantifier

**Definition:** The universal quantification of P(x) is the statement

 $\forall x P(x) = "P(x)" \text{ for all values of } x \text{ in the domain"}$ 

 $\forall$  is called the universal quantifier and  $\forall x P(x)$  is read as "for every x P(x)".

An element for which  $\forall x P(x)$  is false is called a **counterexample** of  $\forall x P(x)$ 

### **Examples:**

$$P(x) = x + 1 > x$$

 $\forall x P(x)$  is **true** where the domain consist of all real numbers.

$$P(x) = x^2 > x$$

 $\forall x P(x)$  is **false** where the domain consist of all real numbers.

 $\forall x P(x)$  is **false** where the domain consist of all integers. (0 is a counterexample.)

## Existential Quantifier

**Definition:** The existential quantification of P(x) is the statement

 $\exists x P(x) =$  "There exists an element x in the domain such that P(x)."

 $\exists$  is called the universal quantifier and  $\exists x P(x)$  is read as "there exists an x such that P(x)".

### **Examples:**

P(x) = x is a prime number greater than  $10^7$ .

 $\exists x P(x)$  is **true** where the domain consist of all integers.

$$P(x) = x = x + 1$$

 $\exists x P(x)$  is **false** where the domain consist of all real numbers.