Lecture 4

Logical Equivalence, Predicates and Quantifiers

Logical Equivalences with $\rightarrow$ and $\leftrightarrow$

- $p \rightarrow q \equiv \neg p \vee q$
- $p \rightarrow q \equiv \neg q \rightarrow \neg p$
- $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
- $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
- $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
- $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

Logical Equivalences

Why logical equivalence laws hold for general compound propositions as well?

Example:

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

$$\overbrace{\neg((r \vee \neg(s \wedge t)) \vee (t \rightarrow (u \leftrightarrow r)))}^{Y} \equiv \overbrace{\neg(r \vee \neg(s \wedge t)) \wedge \neg(t \rightarrow (u \leftrightarrow r))}^{Z}$$

Suppose for $r = T, s = F, t = T$, and $u = T$:

- Truth value of Y and Z differs.
- $(r \vee \neg(s \wedge t))$ has truth value, say W, and $(t \rightarrow (u \leftrightarrow r))$ has truth value, say X.

implies → $\neg(W \vee X) \neq \neg W \wedge \neg X$ *(not possible due to De Morgan's law)*

Proving Logical Equivalences

Proving logical equivalence of two propositions using truth table is **time-consuming**.

Logical equivalence can also be proven using existing laws of logical equivalences.

Example: Prove $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \textit{(by De Morgan's law)} \\
&\equiv \neg p \wedge (\neg(\neg p) \vee \neg q) && \textit{(by De Morgan's law)} \\
&\equiv \neg p \wedge (p \vee \neg q) && \textit{(by double negation law)} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \textit{(by distributive law)} \\
&\equiv F \vee (\neg p \wedge \neg q) && \textit{(by negation law)} \\
&\equiv (\neg p \wedge \neg q) \vee F && \textit{(by commutative law)} \\
&\equiv \neg p \wedge \neg q && \textit{(by identity law)}
\end{aligned}
$$

Predicate Logic

Let's revisit an old example.

Assumptions:
1. All men are mortal.
2. Socrates is a man.

Conclusion: Socrates is mortal.

Does the conclusion follow from assumptions using rules of propositional logic? No, it seems.

We need more powerful form of logic called **predicate logic** to express these reasonings.

Predicates

Statement "x is greater than 3" has two parts.

Subject of the statement: "x"

Predicate of the statement: "is greater than 3"

(Refers to a property that subject can have.)

Convention:

$P(x)$ = "x is greater than 3", where P denotes the predicate and x denotes the subject.

$P(x)$ is called **propositional function** and becomes a proposition when x is assigned a value.

Example:

$Q(x, y) = x$ is a factor of y

$Q(4,100) =$ has truth value 'true', while $Q(3,25)$ has truth value 'false'.

Quantifiers

Quantification expresses the extent to which a predicate is true over a range of elements.

Universal Quantification:

For $P(x)$, it conveys that $P(x)$ is true for all values of x from a certain domain.

Existential Quantification:

For $P(x)$, it conveys that $P(x)$ is true for some value of x from a certain domain.

Domain is important.

Universal Quantifier

Definition: The **universal quantification** of $P(x)$ is the statement

$$\forall x P(x) = \text{“}P(x) \text{ for all values of } x \text{ in the domain”}$$

$\forall$ is called the universal quantifier and $\forall x P(x)$ is read as “for every x $P(x)$”.

An element for which $\forall x P(x)$ is false is called a **counterexample** of $\forall x P(x)$

Examples:

$P(x) = x + 1 > x$

$\forall x P(x)$ is **true** where the domain consist of all real numbers.

$P(x) = x^2 > x$

$\forall x P(x)$ is **false** where the domain consist of all real numbers.

$\forall x P(x)$ is **false** where the domain consist of all integers. *(0 is a counterexample.)*

Existential Quantifier

Definition: The **existential quantification** of $P(x)$ is the statement

$$\exists x P(x) = \text{“There exists an element } x \text{ in the domain such that } P(x).\text{”}$$

$\exists$ is called the universal quantifier and $\exists x P(x)$ is read as “there exists an x such that $P(x)$”.

Examples:

$P(x)$ = x is a prime number greater than 10^7.

$\exists x P(x)$ is **true** where the domain consist of all integers.

$P(x)$ = $x = x + 1$

$\exists x P(x)$ is **false** where the domain consist of all real numbers.